In this derivation, two objects, with masses \(m_1\) and \(m_2\), collide in a completely inelastic collision.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(v_{f1} = v_{f2} = v_f) final velocities equal</td>
</tr>
<tr>
<td>2.</td>
<td>(m_1v_{f1} + m_2v_{f2} = m_1v_{f1} + m_2v_{f2}) conservation of momentum</td>
</tr>
<tr>
<td>3.</td>
<td>(v_f(m_1 + m_2) = m_1v_{f1} + m_2v_{f2}) factor out (v_f)</td>
</tr>
<tr>
<td>4.</td>
<td>(v_f = \frac{m_1v_{f1} + m_2v_{f2}}{m_1 + m_2}) divide</td>
</tr>
</tbody>
</table>

Example 1 on the right applies this equation to a collision seen on many fall weekends: a football tackle.

Center of mass: Average location of mass. An object can be treated as though all its mass were located at this point.

The center of mass is useful when considering the motion of a complex object, or system of objects. You can simplify the analysis of motion of such an object, or system of objects, by determining its center of mass. An object can be treated as though all its mass is located at this point. For instance, you could consider the force of a weightlifter lifting the barbell pictured above as though she applied all the force at the center of mass of the bar, and determine the acceleration of the center of mass.

You may react: “But we have been doing this in many sections of this book,” and yes, implicitly we have been. If we asked earlier how much force was required to accelerate this barbell, we assumed that the force was applied at the center of mass, rather than at one end of the barbell, which would cause it to rotate.

In this section, we focus on how to calculate the center of mass of a system of objects. Consider the barbell above. Its center of mass is on the rod that connects the two balls, nearer the ball labeled “Work,” because that ball is more massive.

When an object is symmetrical and made of a uniform material, such as a solid sphere of steel, the center of mass is at its geometric center. So for a sphere, cube or other symmetrical shape made of a uniform material, you can use your sense of geometry and decide where the center of the object is. That point will be the center of the mass. (We can relax the condition of uniformity if an object is composed of different parts, but each one of them is symmetrical, like a golf ball made of different substances in spherically symmetrical layers.)

The center of mass does not have to lie inside the object. For example, the center of mass of a doughnut lies in the middle of its hole.

The equation to the right can be used to calculate the overall center of mass of a set of objects whose individual centers of mass lie along a line. To use the equation, place the center of mass of each object on the x axis. It helps to choose for the origin a point where one of the centers of mass is located, since this will simplify the calculation. Then, multiply the mass of each object times its center’s x position and divide the sum of these products by the sum of the masses. The resulting value is the x position of the center of mass of the set of objects.
If the objects do not conveniently lie along a line, you can calculate the \( x \) and \( y \) positions of the center of mass by applying the equation in each dimension separately. The result is the \( x, y \) position of the system’s center of mass.

\[
x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}
\]

- \( x_{CM} \) = \( x \) position of center of mass
- \( m_i \) = mass of object \( i \)
- \( x_i \) = \( x \) position of object \( i \)

**What is the location of the center of mass?**

\[
x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}
\]

\[
x_{CM} = \frac{(3.0 \text{ kg})(0.0 \text{ m}) + (2.0 \text{ kg})(16 \text{ m})}{3.0 \text{ kg} + 2.0 \text{ kg}}
\]

\[
x_{CM} = \frac{32}{5.0} = 6.4 \text{ m}
\]

7.15 - Center of mass and motion

**Center of mass and motion**

Laws of mechanics apply to center of mass.

Shifting center of mass creates "floating" illusion

Above, you see a ballet dancer performing a grand jeté, a "great leap." When a ballet dancer performs this leap well, she seems to float through the air. In fact, if you track the dancer’s motion by noting the successive positions of her head, you can see that its path is nearly horizontal. She seems to be defying the law of gravity. This seeming physics impossibility is explained by considering the dancer’s center of mass. An object (or a system of objects) can be analyzed by considering the motion of its center of mass.

Look carefully at the locations of the dancer’s center of mass in the diagram. The center of mass follows the parabolic path of projectile motion.

To achieve the illusion of floating – moving horizontally – the dancer alters the location of her center of mass relative to her body as she performs the jump. As she reaches the peak of her leap, she raises her legs, which places her center of mass nearer to her head. This
decreases the distance from the top of her head to her center of mass, so her head does not rise as high as it would otherwise. This allows her head to move in a straight line while her center of mass moves in the mandatory parabolic projectile arc.

At the right, we use another example to make a similar point. A cannonball explodes in midair. Although the two resulting fragments move in different directions, the center of mass continues along the same trajectory the cannonball would have followed had it not exploded. The two fragments have different masses. The path of the center of mass is closer to the path of the more massive fragment, as you might expect.

7.16 - Interactive summary problem: types of collisions

On the right is a simulation featuring three collisions. Each collision is classified as one of the following: an elastic collision, a completely inelastic collision, an inelastic (but not completely inelastic) one, or an impossible collision that violates the laws of physics. The colliding disks all have the same mass, and there is no friction. Each disk on the left has an initial velocity of 1.00 m/s. The disks on the right have an initial velocity of −0.60 m/s.

Press GO to watch the collisions. Use the PAUSE button to stop the action after the collisions and record data, then make whatever calculations you need to classify each collision using the choices in the drop-down controls labeled “Collision type.” Press RESET if you want to start the simulation from the beginning.

If you have difficulty with this, review the sections on elastic and inelastic collisions.

7.17 - Gotchas

One object has a mass of 1 kg and a speed of 2 m/s, and another object has a mass of 2 kg and a speed of 1 m/s. The two objects have identical momenta. Only if they are moving in the same direction. You can say they have equal magnitudes of momentum, but momentum is a vector, so direction matters. Consider what happens if they collide. The result will be different depending on whether they are moving in the same or opposite directions.

In inelastic collisions, momentum is not conserved. No. Kinetic energy decreases, but momentum is conserved.

Two objects are propelled by equal constant forces, and the second one is exposed to its force for three times as long. The second object’s change in momentum must be greater than the first’s. This is true. It experienced a greater impulse, and impulse equals the change in momentum.
Step-by-step solution

We use the fact that the torques sum to zero to solve this problem. There are only two torques in the problem due to the location of the axis of rotation: The tension creates no torque.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>torque of witch + torque of duck = 0</td>
</tr>
<tr>
<td>2.</td>
<td>[-(0.183 \text{ m})(m_w g)+(1.65 \text{ m})(44.5 \text{ N})=0 \text{ N} \cdot \text{m}]</td>
</tr>
<tr>
<td>3.</td>
<td>[m_w g = (1.65 \text{ m})(44.5 \text{ N}) / (0.183 \text{ m})]</td>
</tr>
<tr>
<td>4.</td>
<td>[m_w g = 401 \text{ N}]</td>
</tr>
</tbody>
</table>

### 11.3 - Center of Gravity

Center of gravity: The force of gravity effectively acts at a single point of an object called the center of gravity.

The concept of center of gravity complements the concept of center of mass. When working with torque and equilibrium problems, the concept of center of gravity is highly useful.

Consider the barbell shown above. The sphere on the left is heavier than the one on the right. Because the spheres are not equal in weight, if you hold the barbell exactly in the center, the force of gravity will create a torque that causes the barbell to rotate. If you hold it at its center of gravity however, which is closer to the left ball than to the right, there will be no net torque and no rotation.

When a body is symmetric and uniform, you can calculate its center of gravity by locating its geometric center. Let’s consider the barbell for a moment as three distinct objects: the two balls and the bar. Because each of the balls on the barbell is a uniform sphere, the geometric center of each coincides with its center of gravity. Similarly, the center of gravity of the bar connecting the two spheres is at its midpoint.

When we consider the entire barbell, however, the situation gets more complicated. To calculate the center of gravity of this entire system, you use the equation to the right. This equation applies for any group of masses distributed along a straight line. To apply the equation, pick any point (typically, at one end of the line) as the origin and measure the distance to each mass from that point. (With a symmetric, uniform object like a ball, you measure from the origin to its geometric center.)

Then, multiply each distance by the corresponding weight, add the results, and divide that sum by the sum of the weights. The result is the distance from the origin you selected to the center of gravity of the system. The center of gravity of an object does not have to be within the mass of the object: For example, the center of gravity of a doughnut is in its hole.

If you have studied the center of mass, you may think the two concepts seem equivalent. They are. When \(g\) is constant across an object, its center of mass is the same as its center of gravity. Unless the object is enormous (or near a black hole where the force of gravity changes greatly with location), a constant \(g\) is a good assumption.

You can empirically determine the center of gravity of any object by dangling it. In Concept 1, you see the center of gravity of a painter’s palette being determined by dangling. To find the center of gravity of an object using this method, hang (dangle) the object from a point and allow it to move until it naturally stops and rests in a state of equilibrium. The center of gravity lies directly below the point where the object is suspended, so you can draw an imaginary line through the object straight down from the point of suspension. The object is then dangled again, and you draw another line down from the suspension point. Since both of these lines go through the center of gravity, the center of gravity is the point where the lines intersect.

\[
x_{\text{CG}} = \frac{w_1 x_1 + w_2 x_2 + \ldots + w_n x_n}{w_1 + w_2 + \ldots + w_n}
\]

- \(x_{\text{CG}}\) = \(x\) position of center of gravity
- \(w_i\) = weight of object \(i\)
- \(x_i\) = \(x\) position of object \(i\)
Where is the center of gravity?

\[ x_{CG} = \frac{w_1 x_1 + w_2 x_2 + \ldots + w_n x_n}{w_1 + w_2 + \ldots + w_n} \]

Put the origin at center of gold ball:

\[ x_{CG} = \frac{(31 \text{ N})(0 \text{ m}) + (12 \text{ N})(0.90 \text{ m}) + (23 \text{ N})(1.3 \text{ m})}{31 + 12 + 23 \text{ N}} \]

\[ x_{CG} = \frac{(8.4 + 29.9)}{66} \]

\[ x_{CG} = 0.58 \text{ m from center of gold-colored ball} \]

11.4 - Interactive problem: achieve equilibrium

In the simulation on the right, you are asked to apply three forces to a rod so that it will be in static equilibrium. Two of the forces are given to you and you have to calculate the magnitude, position, and direction of the third force. If you do this correctly, when you press GO, the rod will not move.

The rod is 2.00 meters long, and is horizontal. A force of 323 N is applied to the left end, straight up. A force of 627 N is applied to the right end, also straight up. You are asked to apply a force to the rod that will balance these two forces and keep it in static equilibrium.

Here is a free-body diagram of the situation. We have not drawn the third force where it should be!

After you calculate the third vector’s magnitude, position and direction, follow these steps to set up the simulation.

1. Adjust the rod length so it is 2.00 m.
2. Drag the axis of rotation to an appropriate position.
3. Apply all the forces. Drag a force vector by its tail from the control panel and attach the tail to the rod. You can then move the tail of the vector along the rod to the correct position, and drag the head of the vector to change its length and angle.

The control panel will show you the force's magnitude, direction and distance to the axis of rotation. The vector whose properties are being displayed has its head in blue.

When you have the simulation set up, press GO. If everything is set up correctly, the rod will be in equilibrium and will not move. Press RESET if you need to make any adjustments. If you have trouble, refer to the section on static equilibrium in this chapter, and the section on torque in the Rotational Dynamics chapter.

After you solve this interactive problem, consider the following additional challenge. What do you think will happen in the simulation if you change the position of the axis of rotation? Make a guess, and test your hypothesis with the simulation.
Section 13 - Inelastic collisions

13.1 Ball A has mass 5.0 kg and is moving at $-3.2 \text{ m/s}$ when it strikes stationary ball B, which has mass 3.9 kg, in a head-on collision. If the collision is elastic, what is the velocity of (a) ball A, and (b) ball B after the collision? (c) If the collision is completely inelastic, what is the common velocity of balls A and B?

(a) \underline{\phantom{0000}} \text{ m/s}

(b) \underline{\phantom{0000}} \text{ m/s}

(c) \underline{\phantom{0000}} \text{ m/s}

13.2 During a snowball fight, two snowballs travelling towards each other collide head-on. The first is moving east at a speed of 16.1 m/s and has a mass of 0.450 kg. The second is moving west at 13.5 m/s. When the snowballs collide, they stick together and travel west at 3.50 meters per second. What is the mass of the second snowball?

\underline{\phantom{0000000000}} \text{ kg}

13.3 Three railroad cars, each with mass $2.3 \times 10^4 \text{ kg}$, are moving on the same track. One moves north at 18 m/s, another moves south at 12 m/s, and third car between these two moves south at 6 m/s. When the three cars collide, they couple together and move with a common velocity. What is their velocity after they couple?

\underline{\phantom{0000000}} \text{ m}

13.4 A 110 kg quarterback is running the ball downfield at 4.5 m/s in the positive direction when he is tackled head-on by a 150 kg linebacker moving at $-3.8 \text{ m/s}$. Assume the collision is completely inelastic. (a) What is the velocity of the players just after the tackle? (b) What is the kinetic energy of the system consisting of both players before the collision? (c) What is the kinetic energy of the system consisting of both players after the collision?

(a) \underline{\phantom{0000000}} \text{ m/s}

(b) \underline{\phantom{0000000000}} \text{ J}

(c) \underline{\phantom{0000000000}} \text{ J}

13.5 Two clay balls of the same mass stick together in an completely inelastic collision. Before the collision, one travels at 5.6 m/s and the other at 7.8 m/s, and their paths of motion are perpendicular. If the mass of each ball is 0.21 kg, what is the magnitude of the momentum of the combined balls after the collision?

\underline{\phantom{000000000}} \text{ kg} \cdot \text{ m/s}

13.6 A large flat 3.5 kg boogie board is resting on the beach. Jessica, whose mass is 55 kg, runs at a constant horizontal velocity of 2.8 m/s. While running, she jumps on the board, and the two of them move together across the beach. (a) What is the speed of the board (with Jessica on it) just after she jumps on? (b) If the board and Jessica slide 7.7 m before coming to a stop, what is the coefficient of kinetic friction between the board and the beach?

(a) \underline{\phantom{0000000}} \text{ m/s}

(b) \underline{\phantom{0000000000}}

13.7 A 6.0 kg ball A and a 5.0 kg ball B move directly toward each other in a head-on collision, then move in opposite directions away from the site of the collision. Ball A has velocity 4.1 m/s before the collision and $-1.1 \text{ m/s}$ after, and ball B has velocity $-2.9 \text{ m/s}$ before the collision. (a) What is ball B's velocity after the collision? (b) Is this an elastic collision?

(a) \underline{\phantom{000000000}} \text{ m/s}

(b) \underline{\phantom{0000000000}} \text{ Yes} \quad \underline{\phantom{0000000000}} \text{ No}

Section 14 - Center of mass

14.1 How far is the center of mass of the Earth-Moon system from the center of the Earth? The Earth's mass is $5.97 \times 10^{24} \text{ kg}$, the Moon's mass is $7.4 \times 10^{22} \text{ kg}$, and the distance between their centers is $3.8 \times 10^8 \text{ m}$.

\underline{\phantom{000000000}} \text{ m}

14.2 Four particles are positioned at the corners of a square that is 4.0 m on each side. One corner of the square is at the origin, one on the positive x axis, one in the first quadrant and one on the positive y axis. Starting at the origin, going clockwise, the particles have masses 2.3 kg, 1.4 kg, 3.7 kg, and 2.9 kg. What is the location of the center of mass of the system of particles?

(\underline{\phantom{00000000}}, \underline{\phantom{00000000}}) \text{ m}
1.8 The sketch shows a mobile in equilibrium. Each of the rods is 0.16 m long, and each hangs from a supporting string that is attached one fourth of the way across it. The mass of each rod is 0.10 kg. The mass of the strings connecting the blocks to the rods is negligible. What is the mass of (a) block A? (b) block B?
(a) ________ kg
(b) ________ kg

1.9 Two identical 1.80 m long boards just barely balance on the edge of a table, as shown in the figure. What is the distance \(x\)?

Section 3 - Center of gravity

3.1 A weightlifter has been given a barbell to lift. One end has a mass of 5.5 kg while the other end has a mass of 4.7 kg. The bar is 0.20 m long. (Consider the bar to be massless, and assume that the masses are thin disks, so that their centers of mass are at the ends of the bar.) How far from the heavier end should she hold the bar so that the weight feels balanced?

3.2 A 0.65 m rod with uniform mass distribution runs along the \(x\) axis with its left end at the origin. A 1.8 m rod with uniform mass distribution runs along the \(y\) axis with its top end at the origin. Find the coordinates of the center of gravity for this system.

(_______, ________ ) m

3.3 A length of uniform wire is cut and bent into the shapes shown. Find the location of the center of gravity of each shape. In each instance, consider the corners of the shape to be located at integer coordinates.

(a) (_______, ________ )
(b) (_______, ________ )
(c) (_______, ________ )
(d) (_______, ________ )

3.4 Three beetles stand on a grid. Two beetles have the same weight, \(W\), and the third beetle weighs \(2W\). (a) The lighter beetles are located at (1.00, 0) and (0, 2.00), and the heavier beetle is at (3.00, 1.00). Find the coordinates of the center of gravity of the beetles. (b) If the heavy beetle moves to (1.00, 1.00), what is the new location of the center of gravity?

(a) (_______, ________ )
(b) (_______, ________ )

3.5 A woman with weight 637 N lies on a bed of nails. The bed has a weight of 735 N and a length of 1.72 m. The bed is held up by two supports, one at the head and one at the foot. Underneath each support is a scale. When the woman lies in the bed, the scale at the foot reads 712 N. How far is the center of gravity of the system from the foot of the bed of nails?

_______ m

3.6 (a) An empty delivery truck weighs \(5.20 \times 10^5\) N. Of this weight, \(3.20 \times 10^5\) N is on the front wheels. The distance between the front axle and the back axle is 4.10 m. How far is the center of gravity of the truck from the front wheels? (b) Now the delivery truck is loaded with a \(2.40 \times 10^5\) N shipment, 2.60 m from the front wheels. Now how far is the center of gravity of the truck from the front wheels?

(a) ________ m
(b) ________ m

3.7 A skateboarder stands on a skateboard so that 62% of her weight is located on the front wheels. If the distance between the